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SOME RESULTS OF MAGIC GRAPHS

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1. We shall consider an undirected finite graph $\mathbf{G} = [V(\mathbf{G}), E(\mathbf{G})]$ without loops or isolated vertices. A graph is *magic* if the edges can be labeled with positive real numbers in such a way that

(i) distinct edges have distinct labels, and
(ii) the sum of the labels of edges incident to each vertex is the same.

A graph is *semimagic* if the labeling with positive numbers satisfies the condition (ii).

The suggestion to study magic graphs was given by *Jiří Sedláček* [4]. Some sufficient conditions for existence of magic graphs are given in [3], [5] and [6]. A characterization of regular magic graphs in terms of even circuits is given in [1]. *J.Mülbacher* [3] used matrix theory to prove two necessary conditions for the existence of magic graphs. These conditions are weaker then in Theorem 2. We denote by $\mathbf{C}_n$ or $\mathbf{P}_n$ the graph which consists of one cycle or one path of length n and by $\mathbf{D}_n$ the totally disconnected graph with n vertices. A spanning subgraph $\mathbf{F}$ of graph $\mathbf{G}$ is called a $\mathbf{F}$-*factor* of $\mathbf{G}$ if its every component is a regular graph of degree 1 or 2. All isolated edges of $\mathbf{F}$ form the linear part $L(\mathbf{F})$ and all cycles form the cyclic part $C(\mathbf{F})$. We say that a $\mathbf{F}$-factor $\mathbf{F}$ *separates* edges e_1 and e_2, if at least of them belongs to $\mathbf{F}$ and neither $L(\mathbf{F})$ and neither $L(\mathbf{F})$ nor $C(\mathbf{F})$ contains both. In [2] the following theorems were proved:

Theorem 1. *A graph is semimagic if and only if its every edge is contained in an* $\mathbf{F}$*-factor.*

Theorem 2. *A graph* $\mathbf{G}$ *is magic if and only if*

(M_1) $\mathbf{G}$ *is a semimagic graphs, and*
(M_2) *every couple of edges of* $\mathbf{G}$ *is separated by an* $\mathbf{F}$*-factor.*

Consequence 1. *If* $\mathbf{C}$ *is magic then there exists a magic labelling of* $\mathbf{G}$ *with positive integer.*

2. So far we do not know the structure of all magic graphs. It is ease to show that graphs $\mathbf{K}_3$, $\mathbf{K}_4$, $\mathbf{K}_{2,2}$, $\mathbf{C}_3 \times \mathbf{P}_1$ and $\mathbf{P}_3 \times \mathbf{P}_1$ (in Figure 1) are not magic, because the couple of edges indicated by dashed lines cannot be separated by an $\mathbf{F}$-factor.

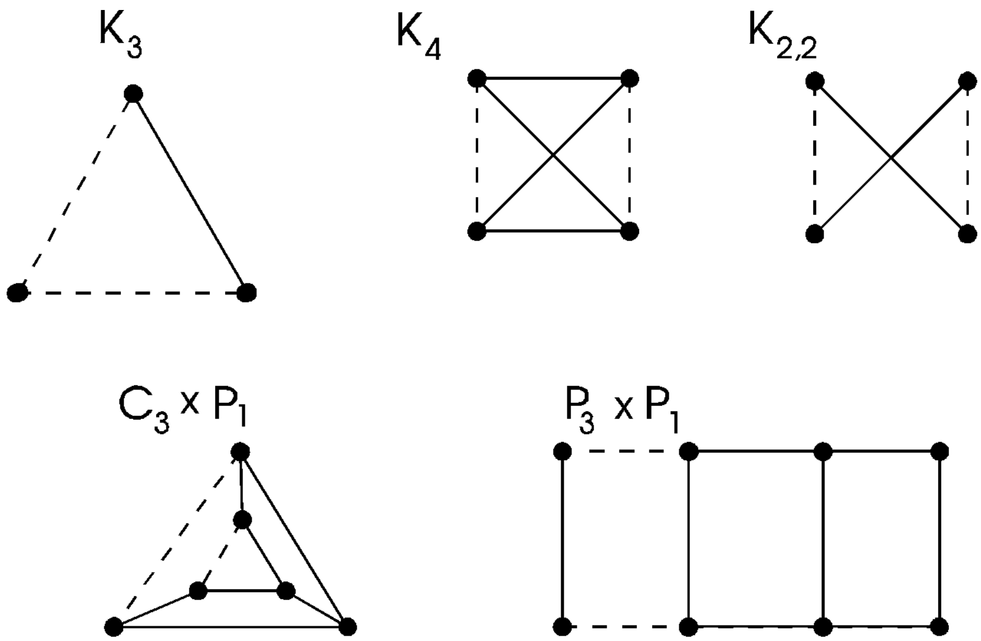

FIGURE 1

Analogous reasoning can be used to prove:

Lemma 1. *A graph* $\mathbf{G}_{2n-1} \times \mathbf{P}_1$ *is not magic for all* $n \geq 2$.

Lemma 2. *A graph* $\mathbf{P}_n \times \mathbf{P}_1$ *is not magic for all* $n \geq 1$.

The following Lemma is a consequence of Theorem 2.

Lemma 3. *A bipartite graph with an odd number of vertices is not magic.*

3. Let $\mathbf{G}^*$ be formed from $\mathbf{G}$ by inserting one new edge. Generally, if $\mathbf{G}$ is magic then $\mathbf{G}^*$ is not magic. For example the graph $\mathbf{P}_3 \times \mathbf{P}_2$ is magic but by adding one edge (indicated by a dashed line in Figure 2) we obtain a non-magic graph.

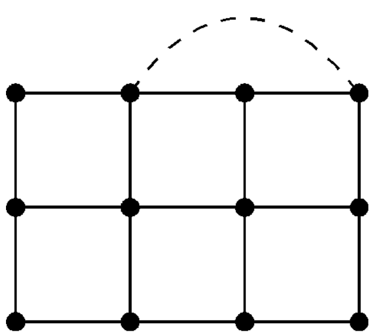

FIGURE 2

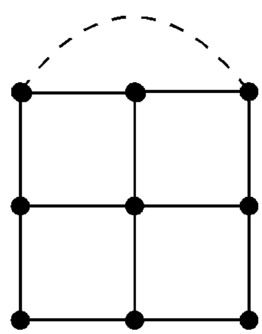

FIGURE 3

On the other hand the graph $\mathbf{P}_2 \times \mathbf{P}_2$ is non-magic.

Theorem 3. *If $\mathbf{G}^*$ is formed from a magic graph $\mathbf{G}$ by inserting a new edge e which belongs to a $\mathbf{F}$-factor of $\mathbf{G}^*$ then $\mathbf{G}^*$ is magic.*

Proof. Evidently every edge of $\mathbf{G}^*$ belongs to a $\mathbf{F}$-factor. If e_i is an arbitrary edge of $\mathbf{G}$ then the couple of edges e, e_i is separated by the $\mathbf{F}$-factor $\mathbf{F}_1$ of $\mathbf{G}$ such that $e_i \in E(\mathbf{F}_1)$. Every couple of edges of $\mathbf{G}$ is separated by $\mathbf{F}$-factor because $\mathbf{G}$ is magic.

4. *M.Doob* [1,p.100] proved the following theorem:

Theorem. *Let $\mathbf{G}$ be a regular graph of degree $d \geq 3$, and $\mathbf{G}_1, \mathbf{G}_2, \ldots, \mathbf{G}_n$ be the connected components of $\mathbf{G}$. Then $\mathbf{G}$ is magic if and only if $\mathbf{G}_i$ is magic, $i = 1, 2, \ldots, n$.*

An analogous statement is not true for non-regular graphs. In Figure 4 there are two magic graphs. (The second of them was shown to the author by *F.X.Steinparz.*) Both have one edge (depicted by dashed lines) which belongs to the cyclic part of all its $\mathbf{F}$-factors.

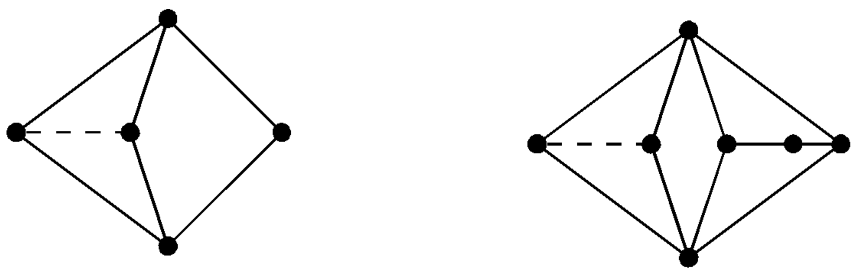

FIGURE 4

In every magic valuation of these graphs this edge must have the value $\frac{r}{2}$ where r is the sum of labels of edges incident to each vertex. The union of these magic graphs is not magic.

Theorem 4. *Let $\mathbf{G}_1, \mathbf{G}_2, \ldots, \mathbf{G}_n$ be the connected components of $\mathbf{G}$. If $\mathbf{G}_i$ is magic, $i = 1, 2, \ldots, n$, and at most one $\mathbf{G}_i$ has one edge which is contained in the cyclic part of all its $\mathbf{F}$-factors then $\mathbf{G}$ is magic.*

The fundamental ideas of the proof are found in proofs of Theorems 5 and 7.

In this section some classes of magic graphs are described.

Theorem 5. *If $\mathbf{G}$ is a semimagic graph none of whose components is $\mathbf{K}_2$ and for every edge $e \in E(\mathbf{G})$ there exists a $\mathbf{F}$-factor $\mathbf{F}$ such that $e \notin C(\mathbf{F})$ then $\mathbf{G} \times \mathbf{P}_1$ is magic.*

Proof. Let v_i, $i = 1, 2, \ldots, |E(\mathbf{G})|$ be the vertices of the semimagic graph $\mathbf{G}$. If $v_i v_j \in E(\mathbf{G})$ we denote by $\mathbf{F}_{i,j}$ one $\mathbf{F}$-factor of $\mathbf{G}$ such that the edge $v_i v_j \in E(\mathbf{F}_{i,j})$. The graph $\mathbf{G} \times \mathbf{P}_1$ consists of $2|V(\mathbf{G})|$ vertices v_i^s, $i = 1, 2, \ldots, |v(\mathbf{G})|$ and $s = 1, 2$, and $(2|E(\mathbf{G}) + v(\mathbf{G})|$ edges $v_i^1 v_i^2$ for all i and $v_i^s v_2^s$, $s = 1$ or 2, if $v_i v_j \in E(\mathbf{G})$.

For every couple e_1, e_2 of edges of $\mathbf{G} \times \mathbf{P}_1$ we describe the $\mathbf{F}$-factor $\mathbf{F}$ which contain the edge e_1 and separates the edges e_1, e_2. We consider four cases.

(a) $e_1 = v_i^s v_j^s$, $e_2 = v_h^s v_k^s$, $s = 1$ or 2 and $(i, j) \neq (h, k)$.

If $e_2 \notin E(\mathbf{F}_{i,j})$ then $\mathbf{F} = \mathbf{F}_{i,j} \cup \mathbf{F}_{i,j}$.

If $e_2 \notin E(\mathbf{F}_{i,j})$ then $\mathbf{F} = \mathbf{F}_{i,j} \cup \mathbf{F}_{i,j} - v_h^1 v_k^1 - v_h^2 v_k^2 + v_h^1 v_h^2 + v_k^1 v_k^2$. (Two edges edges $v_h^1 v_k^1$ and $v_h^2 v_k^2$ are omitted from the union of two graphs $\mathbf{F}_{i,j}$ and two edges $v_h^1 v_h^2$ and $v_k^1 v_k^2$ are added to it.)

(b) $e_1 = v_i^1 v_i^2$, $e_2 = v_h^s v_k^s$, $s = 1$ or 2.
Then the separating $\mathbf{F}$-factor is the graph $\mathbf{D}_{|E(\mathbf{G})|} \times \mathbf{P}_1$.

(c) $e_1 = v_i^1 v_i^2$, $e_2 = v_j^1 v_j^2$, $i \neq j$.
Let $v_i v_k \in E(\mathbf{G})$, $k \neq j$ then $\mathbf{F} = \mathbf{F}_{i,k} \cup \mathbf{F}_{i,k} - v_i^1 v_k^1 - v_i^2 v_k^2 + v_i^1 v_i^2 + v_k^1 v_k^2$.

(d) $e_1 = v_i^1 v_j^1$, $e_2 = v_h^2 v_k^2$.
If $v_i v_j$ and $v_h v^k$ are different edges of $\mathbf{G}$ then $\mathbf{F}$ is the same as in case (a) which separates the edges $e_i^1 v_j^1$ and $v_h^1 v_k^1$. In the opposite case we consider two subcases. If $v_i v_j \in C(\mathbf{F}_{i,j})$ then we choose a $\mathbf{F}$-factor $\mathbf{F}^*$ such that $v_i v_k \in C(\mathbf{F}^*)$ and then $\mathbf{F} = \mathbf{F}_{i,j} \cup \mathbf{F}^*$. If $v_i v_j \in L(\mathbf{F}_{i,j})$ and let $v_i v_j \in E(\mathbf{G})$ for $i \neq j$ then $\mathbf{F} = \mathbf{F}_{i,j} \cup \mathbf{F}_{i,k}$.

Theorem 6. *If $\mathbf{G}$ is a semimagic graph none of whose components is $\mathbf{K}_2$ and $\mathbf{H}$ is a graph every one of whose connected components has at least 3 vertices then $\mathbf{G} \times \mathbf{H}$ is magic.*

The proof is analogous as in Theorem 5.

Example 1. A graph $\mathbf{G}_n \times \mathbf{P}_n$ is magic if and only if $4 \leq n \equiv 0 \pmod 2$ and $m = 1$ or $n \geq 3$ and $m \geq 2$.

The Zykovian product of graph $\mathbf{G}$ and $\mathbf{H}$ is the graph $\mathbf{G} \otimes \mathbf{H}$ such that

$$V(\mathbf{G} \otimes \mathbf{H}) = V(\mathbf{G}) \cup V(\mathbf{H})$$

and

$$E(\mathbf{G} \otimes \mathbf{H}) = E(\mathbf{G} \cup \mathbf{H}) \otimes \{(u,v) \quad \text{for all} \quad u \in V(\mathbf{G}) \quad \text{and} \quad v \in V(\mathbf{H})\}$$

Theorem 7. *If $\mathbf{G}$ is a semimagic graph none of whose components is $\mathbf{K}_2$ or $\mathbf{K}_3$ then $\mathbf{G} \otimes \mathbf{D}_1$ is magic.*

Proof. Let $v_1, v_2, \ldots, v_{|v(\mathbf{G})|}$ are the vertices of $\mathbf{G}$ and u is the vertex of $\mathbf{D}_1$. Let $\mathbf{F}_{i,j}$ be one $\mathbf{F}$-factor of $\mathbf{G}$ such that $v_i v_j \in E(\mathbf{F}_{i,j})$. For every edge $v_h v_k$ of $\mathbf{G}$ which belongs to $E(\mathbf{F}_{i,j})$ we denote by $\mathbf{F}_{i,j}^{h,k}$ the graph

$$\mathbf{F}_{i,j} + v_h u + v_k u \qquad \text{if} \quad v_h v_k \in L(\mathbf{F}_{i,j})$$

or

$$\mathbf{F}_{i,j} + v_h u + v_k u - v_h v_k \qquad \text{if} \quad v_h v_k \in C(\mathbf{F}_{i,j})$$

The edge $v_i u$, $1 \leq i \leq |V(\mathbf{G})$ belongs to $E(\mathbf{F}_{i,j}^{i,j})$ where $v_i v_j \in E(\mathbf{G})$. Every edge $v_i v_j$ belongs to $E(\mathbf{F}_{i,j}^{h,k})$ where $v_h v_k$ is another edge of $E(\mathbf{F}_{i,j})$. The couple of different edges $v_i v_j$, $v_h v_k$ is separated by the $\mathbf{F}$-factor $\mathbf{F}_{i,j}^{h,k}$ if $v_i v_j \in L(\mathbf{F}_{i,j})$ or by $\mathbf{F}_{i,j}^{s,t}$ if $v_i v_j \in C(\mathbf{F}_{i,j})$ and $v_h v_k \notin E(\mathbf{F}_{i,j})$ where $v_s v_t$ is one edge of $\mathbf{F}_{i,j}$ different from $v_i v_j$ or by $\mathbf{F}_{i,j}^{i,j}$ in all other cases.

We assume that $v_h v_k \notin E(\mathbf{G})$. The couple of edges $v_i v_j$, $v_i u$ is separated by $\mathbf{F}_{h,k}^{h,k}$ if $v_i v_j \notin C(\mathbf{F}_{h,k})$ or by $\mathbf{F}_{h,k}^{i,s}$ where $v_i v_s \in C(\mathbf{F}_{h,k})$ and $s \neq u$ in the opposite case.

The couple of different edges $v_i u$, $v_j u$ is separated by $\mathbf{F}_{i,k}^{i,k}$ where $v_i v_k \in E(\mathbf{G})$) and $k \neq j$.

Example 2. (Stewart [6,p.1046]) A complete graph $\mathbf{K}_n$ is magic if and only if $n = 3$ or $n \geq 5$.

Example 3. (Stewart [6,p.1052]) A wheel $\mathbf{W}_n$ is magic if and only if $n \geq 4$.

It is easy to show that:

Lemma 4. *A graph* $\mathbf{P}_n \times \mathbf{P}_m$ *is magic if and only if* $2 \leq n \leq m$ *and* $m, n \equiv 1 \pmod 2$.

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